Cusp effects in meson decays

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Outline

Introduction

- Chiral perturbation theory, the pion mass, and $\pi\pi$ scattering

Non-relativistic effective theory for $K \to 3\pi$ decays

- Cusp phenomenology and heuristic explanation
- NREFT vs. ChPT
- Lagrangians, power counting, matching and all that

Radiative corrections

More of the same? K_L, η, η'

Conclusions

in collaboration with M. Bissegger, G. Colangelo, A. Fuhrer,
J. Gasser, A. Rusetsky, S.P. Schneider

Quark mass expansion of the pion mass (1)

Chiral perturbation theory (ChPT): Weinberg 1966; Gasser, Leutwyler 1984

- chiral symmetry strongly constrains properties and interactions of would-be Goldstone bosons, the pions
- systematic expansion of low-energy observables in terms of (small) quark masses and (small) momenta

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Gell-Mann–Oakes–Renner relation for the pion mass M:

$$M^2 = B(m_u + m_d) \qquad B = -\frac{\langle 0|\bar{u}u|0\rangle}{F^2}$$

- $B \neq 0$ is sufficient (but not necessary) condition for chiral symmetry breaking
- B is an order parameter of chiral symmetry breaking

Quark mass expansion of the pion mass (2)

- at next-to-leading order ($\mathcal{O}(p^4)$, one-loop):

$$M_\pi^2 = M^2 - \frac{M^4}{32\pi^2 F^2}\bar{\ell}_3 + \mathcal{O}(M^6)$$

- correction to Gell-Mann–Oakes–Renner relation:

$$M_\pi^2 = B(m_u + m_d) + A(m_u + m_d)^2 + \mathcal{O}(m_q^3)$$

- how do we know that the leading term dominates?
  what if $\bar{\ell}_3$ is anomalously large?
  $\Rightarrow$ "generalised ChPT", different power counting
  Knecht, Moussallam, Stern, Fuchs 1995

- where can we learn something about $\bar{\ell}_3$?

$\pi\pi$ scattering at next-to-leading order

- $I = 0\ \pi\pi$ scattering length at $\mathcal{O}(p^4)$:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\Big\{1 + \epsilon + \mathcal{O}(M_\pi^4)\Big\}$$

$$\epsilon = \frac{5M_\pi^2}{84\pi^2 F_\pi^2}\left(\bar{\ell}_1 + 2\bar{\ell}_2 - \frac{3}{8}\,\bar{\ell}_3 + \frac{21}{10}\,\bar{\ell}_4 + \frac{21}{8}\right)$$

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&= \frac{M_\pi^2}{3}\langle r^2\rangle_\pi^S + \frac{200\pi}{7}F_\pi^2 M_\pi^2\left(a_2^0 + 2a_2^2\right) - \frac{M_\pi^2}{672\pi^2 F_\pi^2}\left(15\bar{\ell}_3 - 353\right)
\end{aligned}
$$

- therefore: all we need to do is to measure $a_0^0 \;\Rightarrow\;$ extract $\bar{\ell}_3$
know how much of M_π^2 is due to linear term in quark masses!
- . . . but how do you actually measure $\pi\pi$ scattering lengths?

Experiments vs. theory on $\pi\pi$ scattering

Experiments:

- reactions on nucleons, e.g. $\pi N \to \pi\pi N$
- $K^+ \to \pi^+\pi^- e^+\nu_e$ (K_{e4}) — BNL-865, NA48/2, KLOE
- pionium (= $\pi^+\pi^-$ atom) lifetime — DIRAC
- cusp in $K^+ \to \pi^0\pi^0\pi^+$ — NA48/2

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Theory prediction:

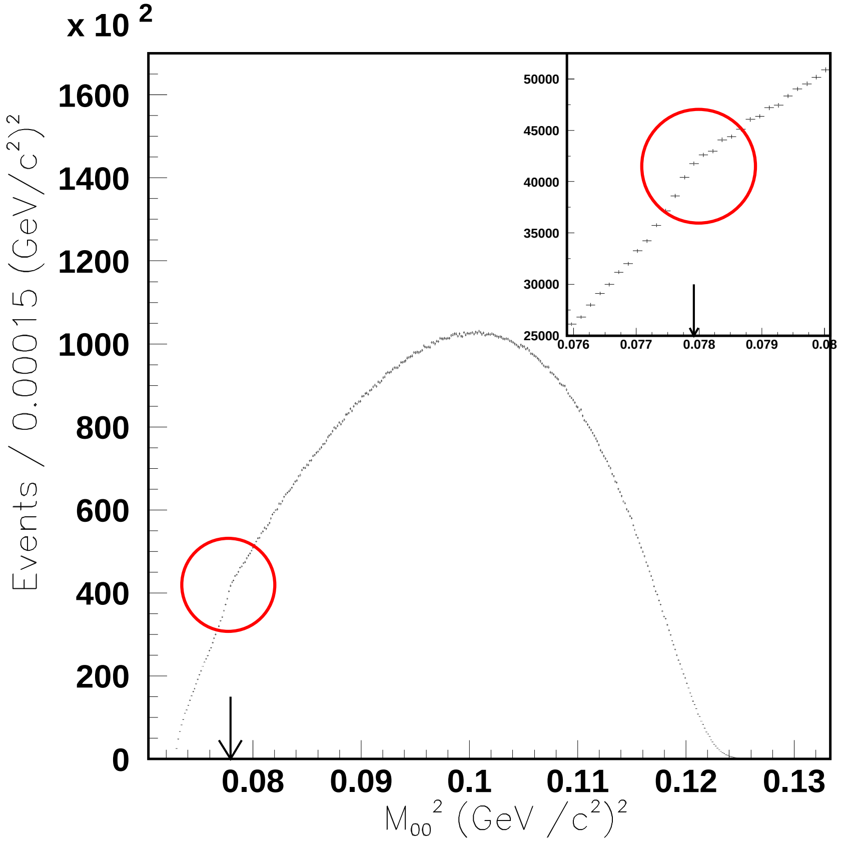
- 2-loop ChPT + Roy equations (dispersion theory):

$$\begin{aligned} a_0 &= 0.220 \pm 0.005 \\ a_2 &= -0.0444 \pm 0.0010 \\ a_0 - a_2 &= 0.265 \pm 0.004 \end{aligned}$$

(for QCD in the isospin limit) $\Rightarrow$ $\simeq 1.5\%$ theoretical precision

Colangelo, Gasser, Leutwyler, PLB 488 (2000) 261, NPB 603 (2001) 125

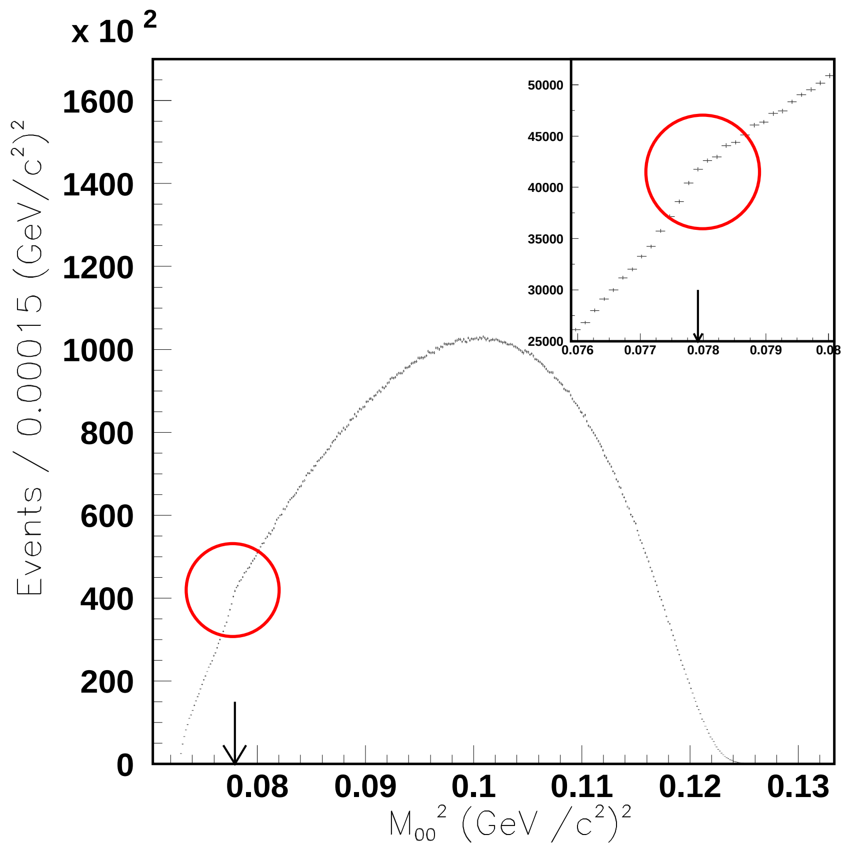
The cusp effect in $K^\pm \to \pi^\pm\pi^0\pi^0$



- cusp at $M_{\pi^0\pi^0} = 2M_{\pi^+}$

Batley et al., PLB 633 (2006) 173

The cusp effect in $K^\pm \to \pi^\pm\pi^0\pi^0$



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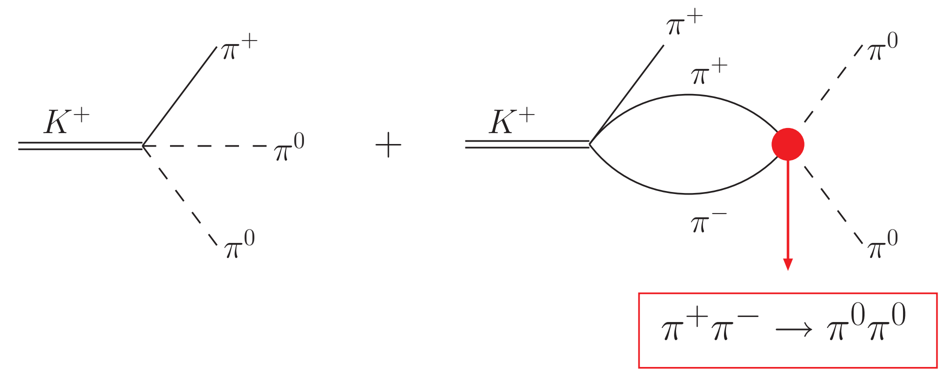

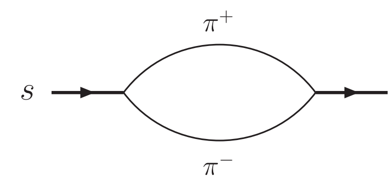

$$= \ldots + \frac{i}{16\pi}\, v_\pm(s)$$

$$v_\pm(s) = \begin{cases} i\sqrt{\dfrac{4M_{\pi^+}^2}{s} - 1}\,, & s < 4M_{\pi^+}^2 \\ \sqrt{1 - \dfrac{4M_{\pi^+}^2}{s}}\,, & s > 4M_{\pi^+}^2 \end{cases}$$

- interference tree + 1-loop below $\pi^+\pi^-$ threshold
- square-root behaviour = cusp

Cabibbo, PRL 93 (2004) 121801

Non-relativistic EFT vs. ChPT

- consider partial waves in $\pi\pi$ scattering:

$$\mathrm{Re}\, T = a + b\, q^2 + c\, q^4 + \ldots$$

scattering length a, effective range b etc.

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- ChPT: a, b, c expanded in powers of M_π^2,

$$a = \frac{7M_\pi^2}{32\pi F_\pi^2} + \mathcal{O}(M_\pi^4)$$

Weinberg 1966

contributions from tree, 1-loop, 2-loop . . .

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  contributions from tree, 1-loop, 2-loop . . .

- NREFT: $a \Leftrightarrow$ tree
  $b \Leftrightarrow$ tree + 2-loop
  $c \Leftrightarrow$ tree + 2-loop + 4-loop
  a, b, c parameters of the theory
  $\Rightarrow$ parametrise T directly in terms of scattering lengths
  $\Rightarrow$ do not predict these, extract as parameters from data

Non-relativistic EFT (1): basics

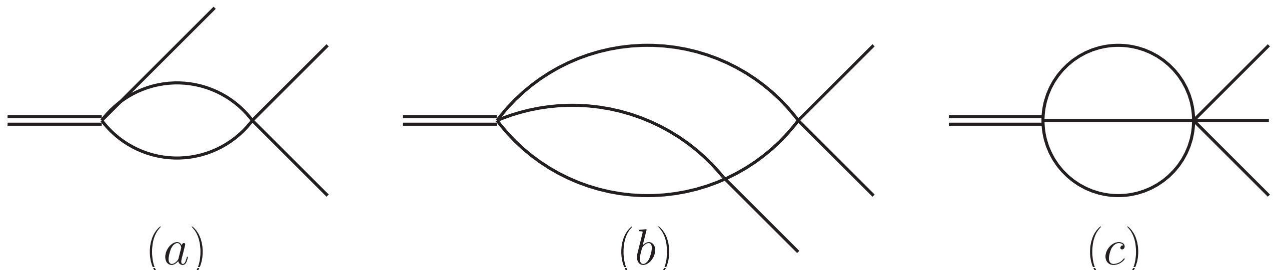
momenta	:	$\lvert\mathbf{p}\rvert/M_\pi = \mathcal{O}(\epsilon)$
kinetic energy	:	$T = \omega(\mathbf{p}) - M_\pi = \mathcal{O}(\epsilon^2)$
in $K \to 3\pi$	:	$M_K - \sum_i M_i = \sum_i T_i = \mathcal{O}(\epsilon^2)$

where $\omega(\mathbf{p}) = \sqrt{M_\pi^2 + \mathbf{p}^2}$

- non-relativistic region = whole decay region (and slightly beyond)
- $\pi\pi$ scattering length small $\Rightarrow$ rescattering perturbative
- two-fold expansion in ϵ and $\pi\pi$ scattering length a
- at given order a, ϵ, only finite number of graphs contribute
  $\Rightarrow$ power counting

Non-relativistic EFT (2): power counting

- organise tree level polynomials in even powers of momenta
  $\Rightarrow \mathcal{O}(\epsilon^0)$, $\mathcal{O}(\epsilon^2)$, $\mathcal{O}(\epsilon^4)$, ...
- loops:



(a) (b) (c)

propagator: $$\frac{1}{\omega(\mathbf{p}) - p^0} = \mathcal{O}(\epsilon^{-2})$$

loop integration: $$d^4p = dp^0 d^3\mathbf{p} = \mathcal{O}(\epsilon^5)$$

- each loop with two-body rescattering $(\epsilon^{-2})^2\epsilon^5 = \mathcal{O}(\epsilon)$ suppressed
  (a) $= \mathcal{O}(a^1\epsilon^1)$
  (b) $= \mathcal{O}(a^2\epsilon^2)$ $\Rightarrow$ correlated expansion in a and ϵ
- loop with three-body rescattering $(\epsilon^{-2})^3(\epsilon^5)^2 = \mathcal{O}(\epsilon^4)$ suppressed
  (c) $= \mathcal{O}(\epsilon^4)$

Non-relativistic EFT (3): Lagrangian

- propagator: $$\underbrace{\frac{1}{M_\pi^2 - p^2}}_{\text{relativistic}} = \underbrace{\frac{1}{2\omega(\mathbf{p})}\frac{1}{\omega(\mathbf{p}) - p^0}}_{\text{"non-relativistic"}} + \underbrace{\frac{1}{2\omega(\mathbf{p})}\frac{1}{\omega(\mathbf{p}) + p^0}}_{\text{antiparticles}}$$

  generated by Lagrangian

  $$\mathcal{L}_{\text{kin}} = \Phi^\dagger (2W)(i\partial_t - W)\Phi \,, \qquad W = \sqrt{M_\pi^2 - \triangle}$$

  Note: non-local $\mathcal{L}_{\text{kin}}$ generates all relativistic corrections; manifestly Lorentz-invariant / frame-independent

- correctly reproduces singularity structure at small momenta $|\mathbf{p}| \ll M_\pi$, subsumes far-away singularities in effective couplings
- interaction terms:

$$\mathcal{L}_{\pi\pi} = C_x\big(\pi_-^\dagger \pi_+^\dagger (\pi_0)^2 + h.c.\big) + \big(\text{derivative terms}\big)$$

$$\mathcal{L}_{K3\pi} = \frac{G_0}{2}\big(K_+^\dagger \pi_+ (\pi_0)^2 + h.c.\big) + \frac{H_0}{2}\big(K_+^\dagger \pi_- (\pi_+)^2 + h.c.\big) + \ldots$$

- Lagrangian-based QFT, analyticity + unitarity obeyed

Matching

- match the $\pi\pi$ coupling constants to the effective range expansion of the $\pi\pi$ scattering amplitude:

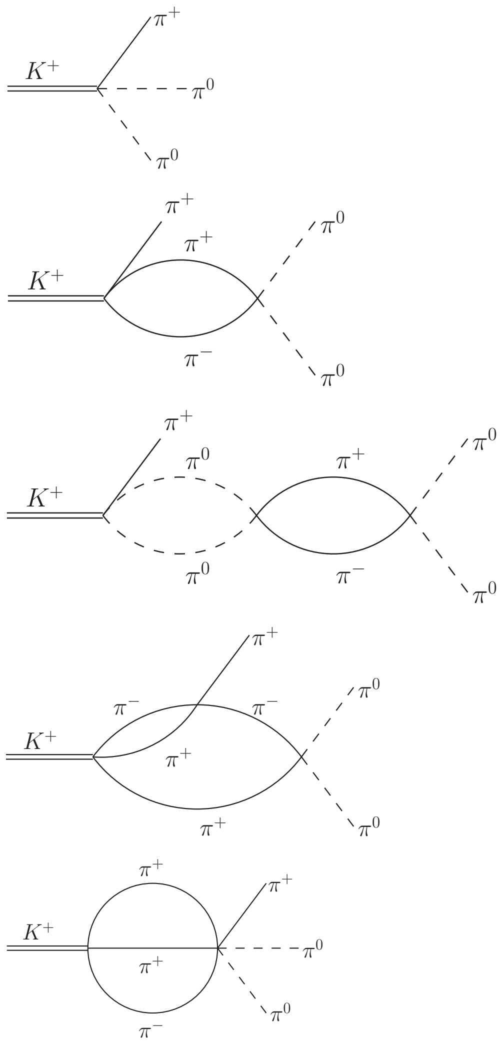
$$\begin{aligned} \text{Re}T_x &= 2C_x + \mathcal{O}(\epsilon^2) \\ 2C_x &= -\frac{32\pi}{3}(a_0 - a_2)\Big\{1 + \underbrace{\frac{M_{\pi^+}^2 - M_{\pi^0}^2}{3M_\pi^2}}_{\text{ChPT } \mathcal{O}(e^2)} + \ldots\Big\} \\ &= -\frac{32\pi}{3}\Big\{a_0 - a_2 + \underbrace{(0.61 \pm 0.16) \times 10^{-2}}_{\text{ChPT } \mathcal{O}(e^2p^2)}\Big\} \end{aligned}$$

Knecht, Urech 1997; Gasser et al. 2001

isospin-breaking corrections in matching calculated in ChPT

- parametrise polynomial $K^+ \to \pi^0\pi^0\pi^+$ in terms of $G_0, G_1, \ldots$
  $K^+ \to \pi^+\pi^+\pi^-$ in terms of $H_0, H_1, \ldots$
- fit all parameters ($G_0, \ldots, H_0, \ldots, C_x, \ldots$) to decays $K^+ \to \pi^0\pi^0\pi^+$ and $K^+ \to \pi^+\pi^+\pi^-$ simultaneously

Representation of $K \longrightarrow 3\pi$ amplitude up to two loops



$$\mathcal{M}^{\text{tree}} = G_0 + G_1(p_3^0 - M_\pi) + \ldots$$

$$\mathcal{M}^{\text{1-loop}} = B_1 J_{+-}(s_3) + B_2 J_{00}(s_3) + \big[B_3 J_{+0}(s_1) + (s_1 \leftrightarrow s_2)\big]$$

$$\mathcal{M}^{\text{2-loop}} = 2G_0 C_x^2 \underbrace{J_{+-}(s_3)J_{00}(s_3)}_{\text{double loops}} + \ldots$$

$$+ \quad 4H_0 C_x C_{+-} \underbrace{F_+(\ldots; s_3)}_{\text{overlapping loops}}$$

$$+ \quad \mathcal{O}(i\,\epsilon^4) \quad [\not\propto \text{ scatt. lengths}]$$

- complete to $\mathcal{O}(\epsilon^4, a\epsilon^5, a^2\epsilon^4)$; valid to all orders in quark masses

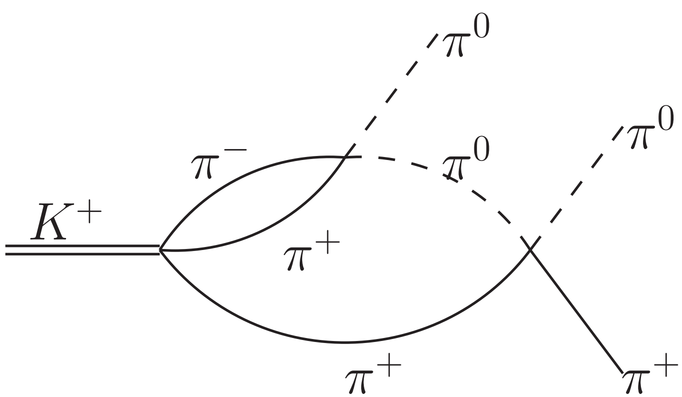
Colangelo, Gasser, BK, Rusetsky, PLB 638 (2006) 187

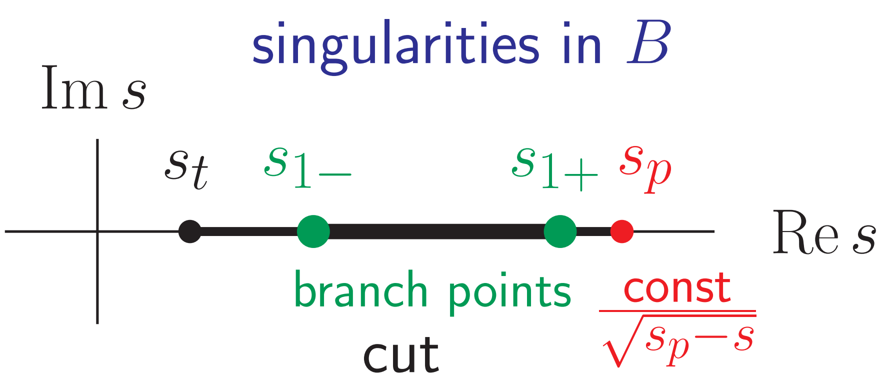
$K \to 3\pi$, analytic properties

- 2-loop function F_+ can be expressed in terms of logarithms;
  close to threshold:

$$F_+(M_{\pi^+}, M_{\pi^+}, M_{\pi^+}, M_{\pi^+}; s) \;=\; \frac{v_{+-}(s)}{256\pi^2}\sqrt{\frac{M_K^2 - 9M_\pi^2}{M_K^2 - M_\pi^2}} = \mathcal{O}(\epsilon^2)$$

- away from threshold, singularity structure is complicated
  approximation $\mathcal{M} = A + Bv_{+-}$ (A, B analytic) does not hold:



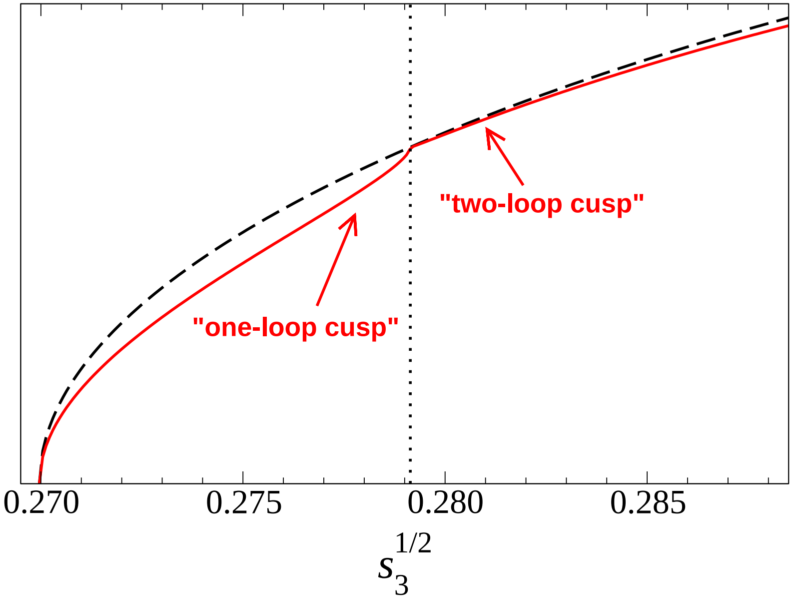


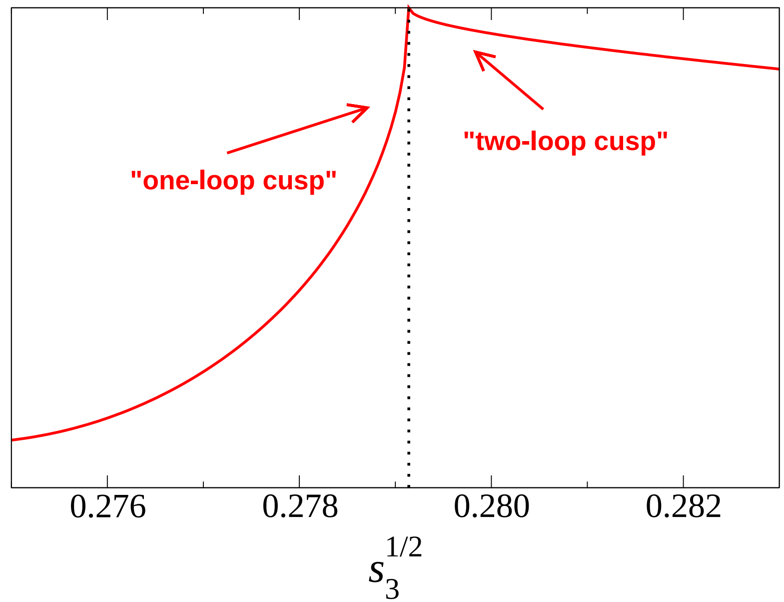
$$\sqrt{s_t} = 275\,\text{MeV}\,, \quad \sqrt{s_{1-}} = 308\,\text{MeV}\,, \quad \sqrt{s_{1+}} = 356\,\text{MeV}\,, \quad \sqrt{s_p} = 359\,\text{MeV}$$

- checked against singularity structure of relativistic loop diagram

Result: cusp(s) at two loops

- cusp at one-loop: $\propto i\, a\, v_{+-}(s) \Rightarrow$ cusp "below" threshold
- cusp at two-loop: $\propto a^2\, v_{+-}(s) \Rightarrow$ cusp "above" threshold





- in principle also sensitive to a_2 via two-loop effects

$$a_0 - a_2 = 0.2815 \pm 0.0045(\text{stat}) \pm \ldots$$
$$a_2 = -0.0693 \pm 0.0136(\text{stat}) \pm \ldots$$

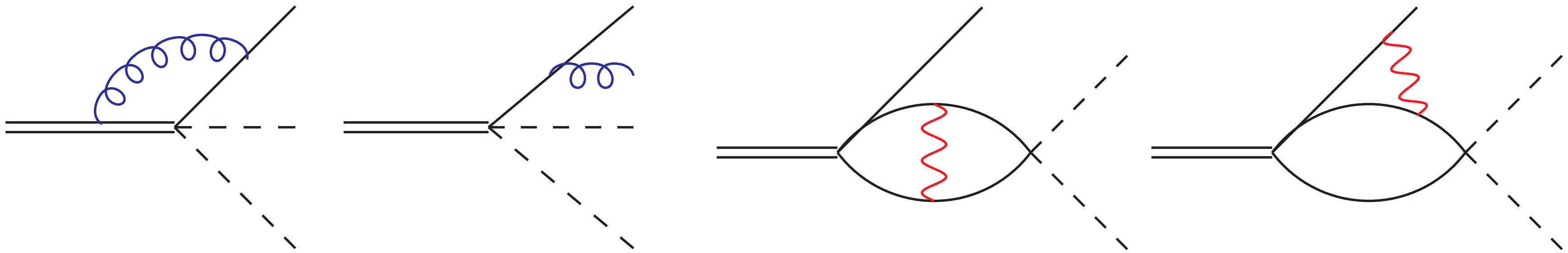
Giudici for NA48/2, Chiral Dynamics (2009)

$\Rightarrow\ a_0 - a_2$ uncomfortably large (by $\approx 3.5\sigma$) – why??

Radiative corrections

Bissegger, Fuhrer, Gasser, BK, Rusetsky, NPB 806 (2009) 178

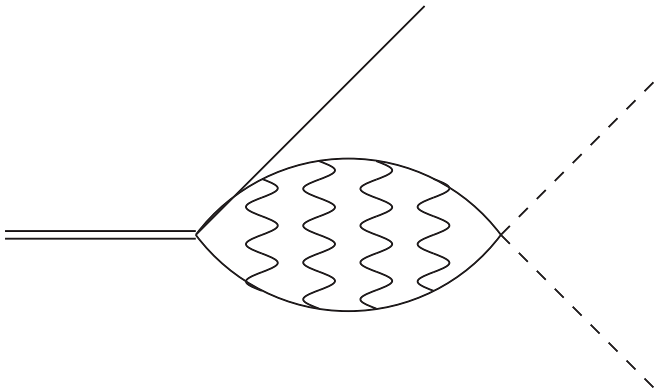
"external" (universal): "internal":



- "external"/universal corrections well-known
  all except Coulomb pole small and smooth
- "internal" corrections with non-trivial effects on threshold behaviour

Non-perturbative effects: pionium

- charged pions may get bound: pionium



ionisation energy: ~ 1.86 keV
ground state width: ~ 0.2 eV

- changes analytical structure at threshold $\Rightarrow$ $\dfrac{\alpha}{v_{\pm}}$ not small

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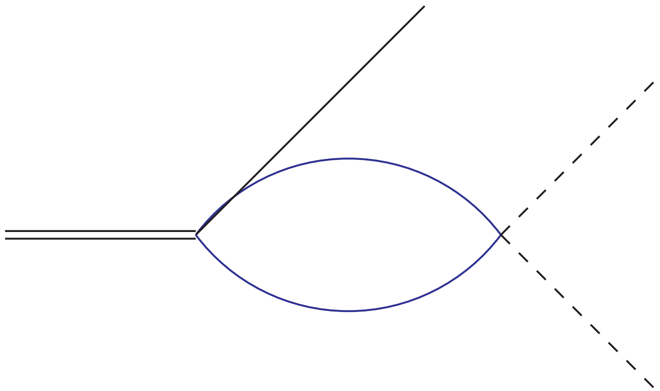

$$G(s) = \frac{i}{16\pi} v_\pm$$

$$v_\pm(s) = \begin{cases} i\sqrt{\frac{4M_{\pi^+}^2}{s} - 1}\,, & s < 4M_{\pi^+}^2\,, \\ \sqrt{1 - \frac{4M_{\pi^+}^2}{s}}\,, & s > 4M_{\pi^+}^2\,, \end{cases}$$

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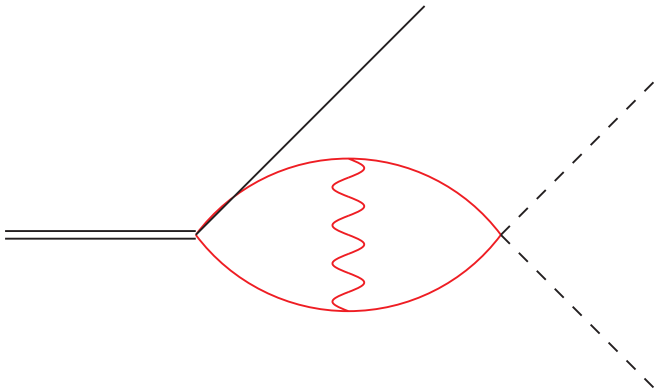

$$G(s) = \frac{i}{16\pi} v_\pm - \frac{\alpha}{32\pi} \Big[\log(-v_\pm^2) \qquad\qquad + C \Big]$$

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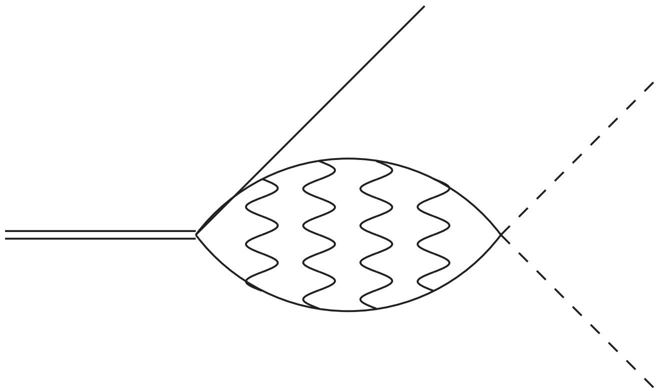

$$G(s) = \frac{i}{16\pi} v_\pm - \frac{\alpha}{32\pi}\left[\log\left(-v_\pm^2\right) + 2\psi\left(1 - \frac{i\,\alpha}{2v_\pm}\right) - 2\psi(1) + C\right]$$

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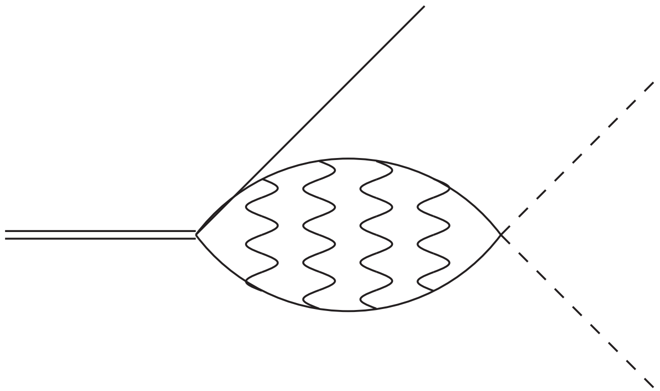


$$G(s) \;=\; \frac{i}{16\pi}v_\pm - \frac{\alpha}{32\pi}\Big[\log\big(-v_\pm^2\big) + 2\psi\Big(1-\frac{i\,\alpha}{2v_\pm}\Big) - 2\psi(1) + C\Big]$$

- one-photon exchange: $\log\big(-v_\pm^2\big)$ divergence at threshold: effect on scattering length surprisingly (?) sizeable:

$$\begin{aligned} a_0 - a_2 &= 0.2571 \pm 0.0048(\text{stat}) \pm 0.0025(\text{syst}) \pm 0.0014(\text{ext}) \\ a_2 &= -0.024 \pm 0.013(\text{stat}) \pm 0.009(\text{syst}) \pm 0.002(\text{ext}) \end{aligned}$$

Giudici for NA48/2, Chiral Dynamics (2009)

$\Rightarrow$ radiative corrections shift $a_0 - a_2$ by $\approx 10\%$!

Combined information on $\pi\pi$ scattering lengths

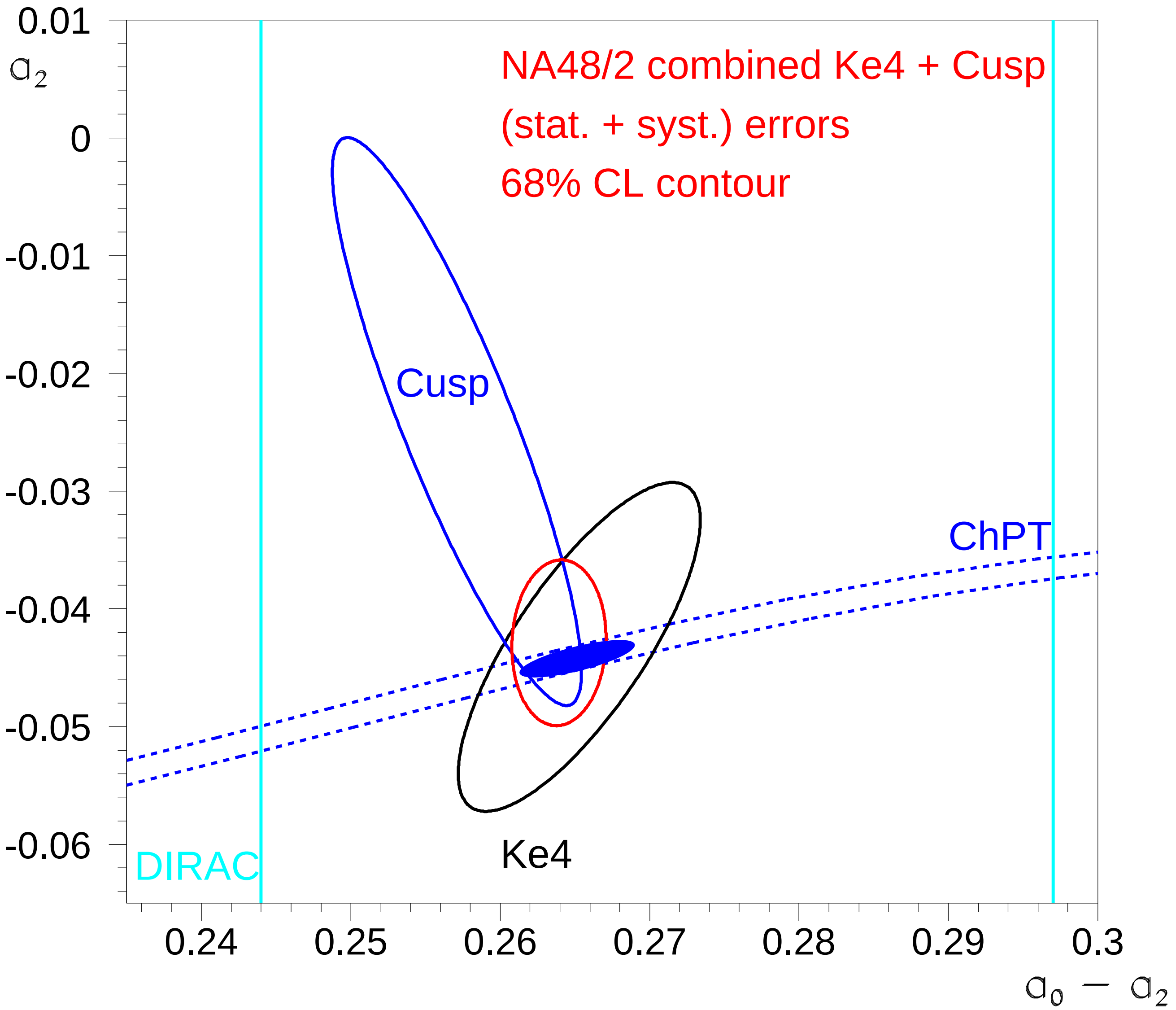
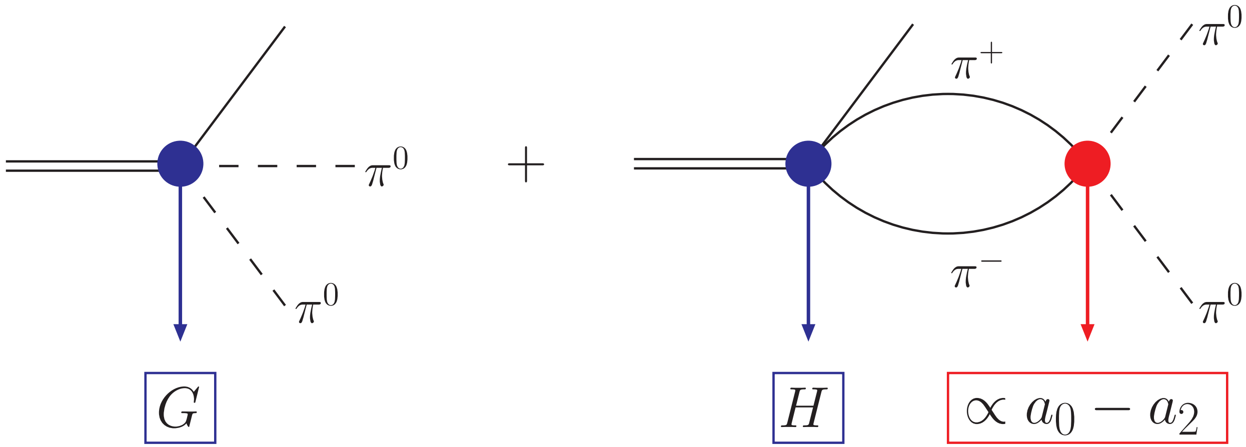


figure courtesy of B. Bloch-Devaux

More of the same? $K_L, \eta \to 3\pi, \eta' \to \eta\pi\pi$

- similar cusps in $K_L \to 3\pi^0$, $\eta \to 3\pi^0$, $\eta' \to \eta\pi^0\pi^0$ ... however:



$\Rightarrow$ cusp strength $\propto H/G \times (a_0 - a_2)$

$\Rightarrow$ sensitivity on $a_0 - a_2$ depends strongly on relative strength "charged" to "neutral" amplitudes

- H/G is strongly different for different channels:

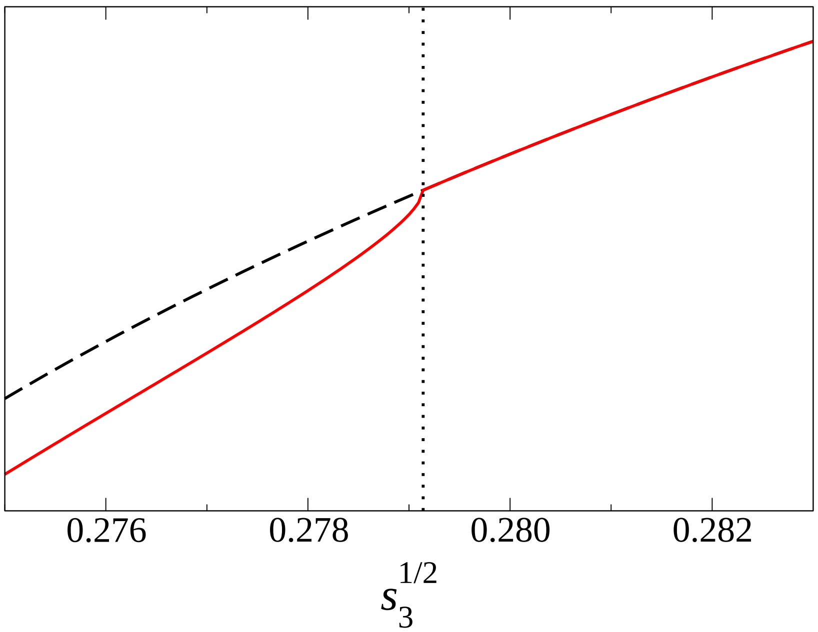
$$\frac{H}{G}(K^+ \to 3\pi) \approx 2 \qquad \frac{H}{G}(K_L \to 3\pi) \approx \frac{1}{3}$$

$$\frac{H}{G}(\eta \to 3\pi) \approx \frac{1}{3} \qquad \frac{H}{G}(\eta' \to \eta\, 2\pi) \approx \sqrt{2}$$

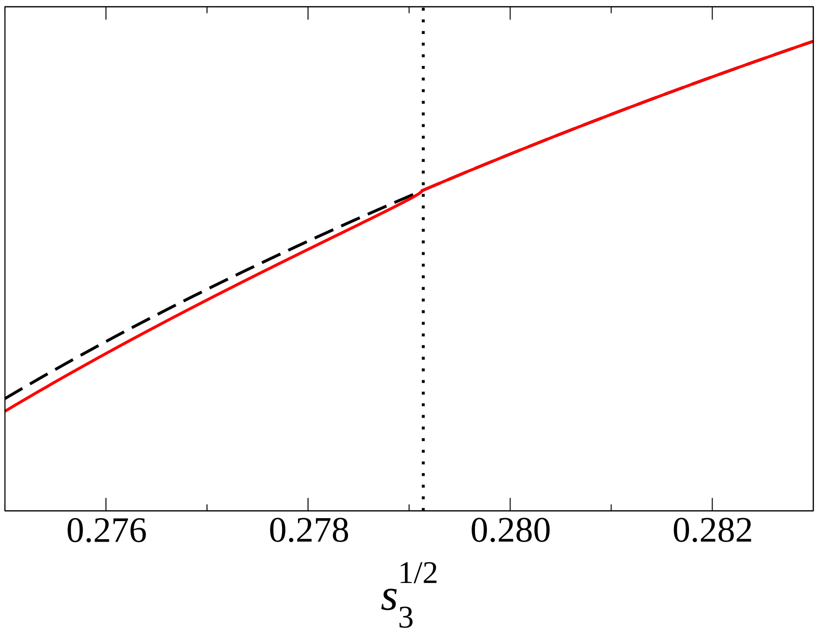
More of the same? $K_L, \eta \to 3\pi, \eta' \to \eta\pi\pi$

- sketches for one-loop cusps:

$K^+ \to \pi^0\pi^0\pi^+$ ($H/G \approx 2$):



$K_L \to 3\pi^0$ ($H/G \approx 1/3$):



- precision determinations of $a_0 - a_2$ much harder from K_L or η
  theoretical formalism exists

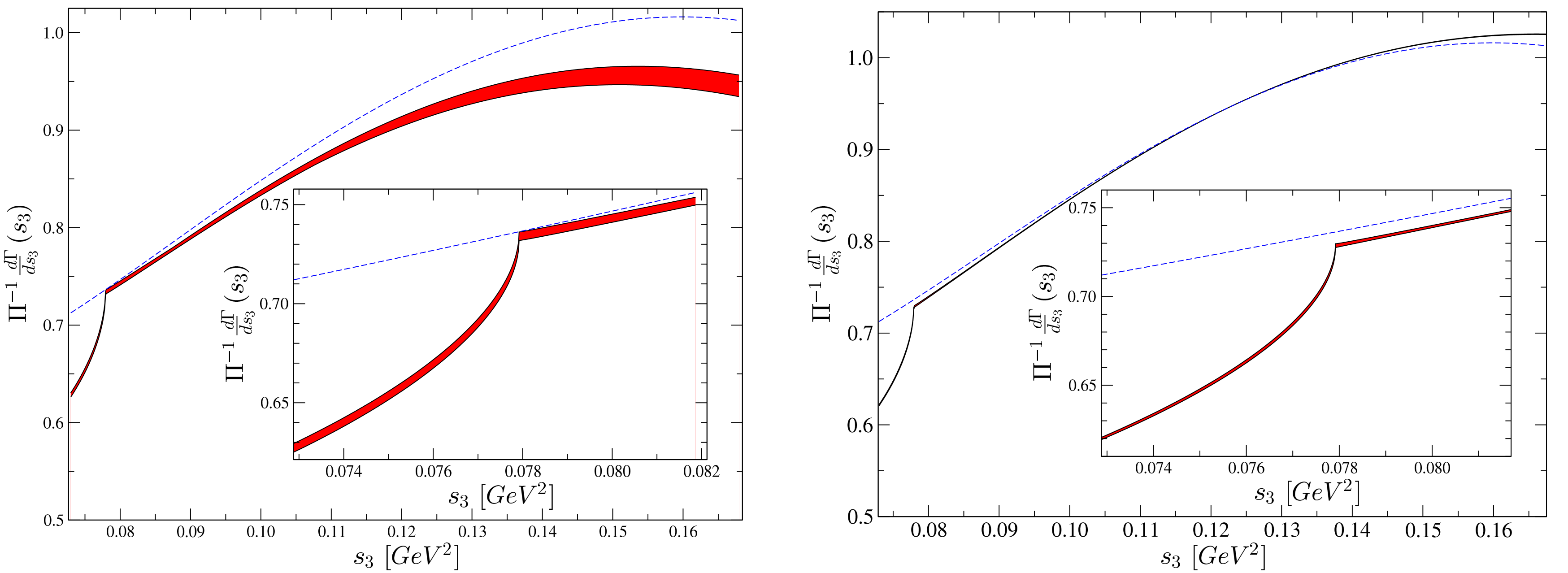
Bissegger et al., PLB 659 (2008) 576, NPB 806 (2009) 178

Gullström, Kupść, Rusetsky, PRC 79 (2009) 028201

Prediction of the cusp in $\eta' \longrightarrow \eta\pi^0\pi^0$

BK, Schneider, EPJC 62 (2009) 511

- new ingredient: $\pi\eta$ scattering in the final state
  not known experimentally, very badly constrained from ChPT
- uncertainty can be absorbed in redefinition of polynomial terms:



- integrated event deficit $\approx 8\%$ below the $\pi^+\pi^-$ threshold

$\Rightarrow$ ELSA, MAMI-C, WASA@COSY, KLOE@DAϕNE, BES-III

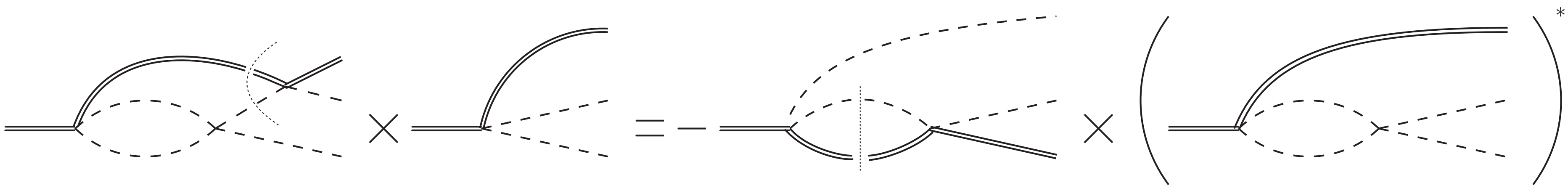
Can one extract the $\pi\eta$ scattering length?

- only possibility to distinguish $\pi\eta$ rescattering from polynomial:

  threshold effects at $s_1 = (M_\pi + M_\eta)^2$

  $\Rightarrow$ border of phase space, cannot go below the cusp

- square-root behaviour above threshold visible? no:
  $\Rightarrow$ interference



2-loop $\times$ tree $= -$ 1-loop $\times$ (1-loop)*

- cusps at $\pi\eta$ threshold cancel exactly
  non-trivial 2-loop graph essential ingredient for this cancellation

Conclusions

- NREFT provides systematic effective field theory framework for an analysis of cusp phenomena and $\pi\pi$ scattering lengths in $K \to 3\pi$ decays
- combined expansion in non-relativistic parameter ϵ
  and scattering lengths a
  currently performed up to $\mathcal{O}(\epsilon^4)$, $\mathcal{O}(a\epsilon^5)$, $\mathcal{O}(a^2\epsilon^4)$
- radiative corrections: additional expansion parameter $e^2 = 4\pi\alpha$
  modification of analytic structure plus
  non-perturbative effects (pionium) in the threshold region
- theoretical accuracy to match experimental one
- similar cusps in $\eta \to 3\pi^0$, $\eta' \to \eta\pi^0\pi^0$

Spares

Different approaches to cusp analysis

- Aim: theoretical framework to extract $\pi\pi$ scattering lengths from experimental data (*not* to predict them!)
- Cabibbo, Isidori, JHEP 0503 (2005) 021:

  analyticity + unitarity + expansion in $\pi\pi$ scattering lengths
  $\Rightarrow$ not fully dynamical scheme (photons?)
  $\Rightarrow$ ansatz $\boxed{\mathcal{M} = A + B\, v_{+-}(s)}$
  A, B analytic, only valid in close vicinity to cusp

- Gámiz, Prades, Scimemi, EPJ C50 (2007) 405:

  Chiral perturbation theory beyond one loop

- Colangelo, Gasser, B.K., Rusetsky, PLB 638 (2006) 187:

  Non-relativistic effective field theory (NREFT)

On the accuracy of the extraction of $a_0 - a_2$

Embarrassing:

$$a_0 - a_2 = 0.264 \pm 0.006(\text{stat}) \pm 0.004(\text{syst}) \pm 0.013(\text{theo})$$

Batley et al., PLB 633 (2006) 173

Sources of theoretical uncertainty:

- electromagnetic corrections $\to$ okay
- isospin breaking in matching $\to$ okay (uncertainty $\lesssim 1\%$)
- three loops, $\mathcal{O}(a^3\epsilon^3)$???

An accurate assessment of the theoretical uncertainty in the extraction of $a_0 - a_2$ can only be worked out along with the data analysis.

Three loops $\mathcal{O}(a^3\epsilon^3)$

Threshold theorem in the absence of photons:

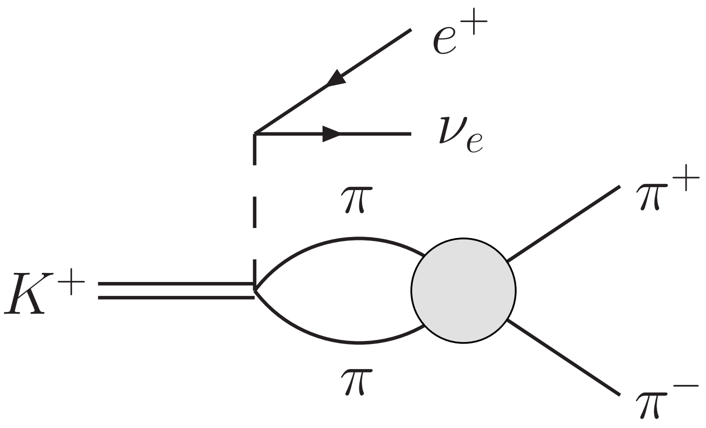
$$\text{coeff}\left[v_{+-}\right] \propto T(K^+ \to \pi^+\pi^+\pi^-)\Big|_{\text{thr}} \times T(\pi^+\pi^- \to \pi^0\pi^0)\Big|_{\text{thr}}$$

- in other words: knowing $T(K^+ \to \pi^+\pi^+\pi^-)$ at $\mathcal{O}(a^n)$ yields leading cusp strength in $T(K^+ \to \pi^0\pi^0\pi^+)$ at $\mathcal{O}(a^{n+1})$
- $T(K^+ \to \pi^+\pi^+\pi^-)$ at two loops $\Rightarrow$ estimate three-loop effect:

$$T(K^+ \to \pi^+\pi^+\pi^-)\Big|_{\text{thr}} \propto -1.0\,(\text{tree}) - 0.13\,i\,(\text{1-loop}) + 0.014\,(\text{2-loop})$$

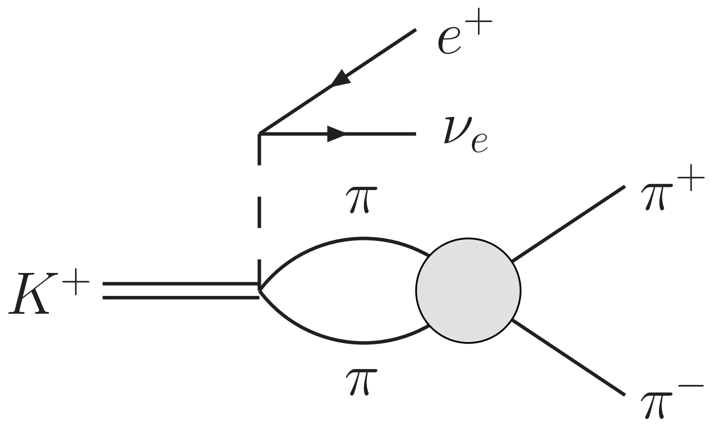
- expect $\mathcal{O}(a^3)$ to modify leading (one-loop) cusp by about 1.5%

K_{e4} decays and $\pi\pi$ scattering



Why can one measure $\pi\pi$ scattering in such a process?

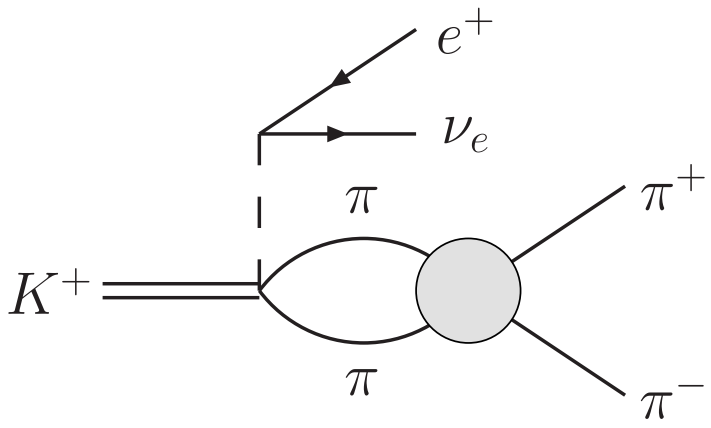
K_{e4} decays and $\pi\pi$ scattering



Why can one measure $\pi\pi$ scattering in such a process?

- Answer:
  - ▷ decay K_{e4} described by form factors
  - ▷ (partial waves of) form factors have phases of $\pi\pi$ interaction (Watson's final state theorem)
  - ▷ measure interference between partial waves $\delta_0^0 - \delta_1^1$

K_{e4} decays and $\pi\pi$ scattering



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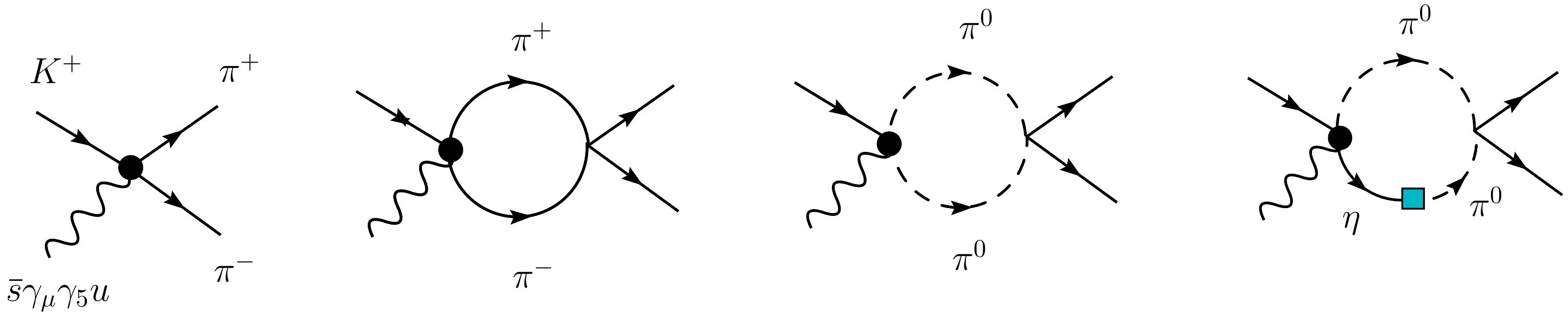
- Problem:
  - ▷ enormous precision of the data NA48/2
  - ▷ what about Watson's theorem and isospin breaking?

Isospin breaking in K_{e4}

Colangelo, Gasser, Rusetsky, EPJC 59 (2009) 777

- two isospin-breaking effects:
  $e \neq 0 \;\Rightarrow\; M_{\pi^+}^2 \neq M_{\pi^0}^2$ and $m_u - m_d$ (e.g. in $\pi^0\eta$ mixing)



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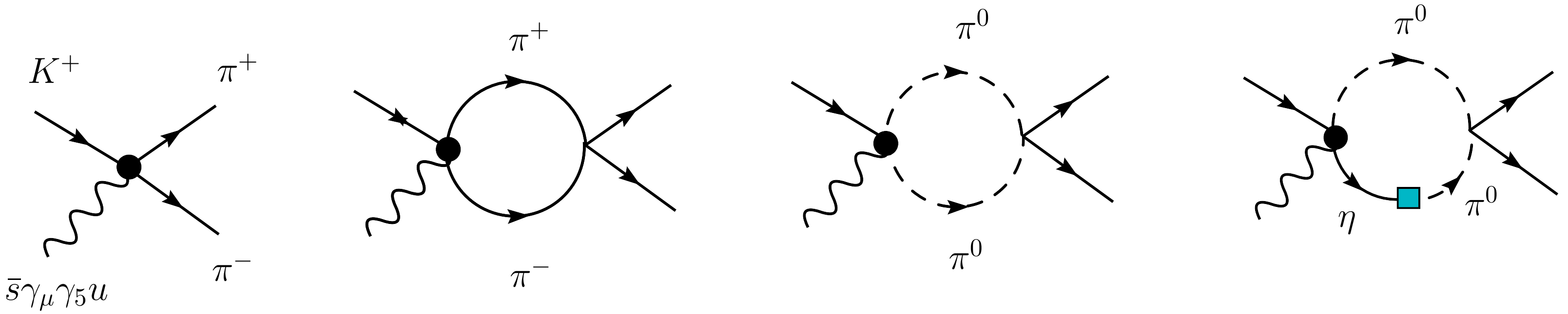
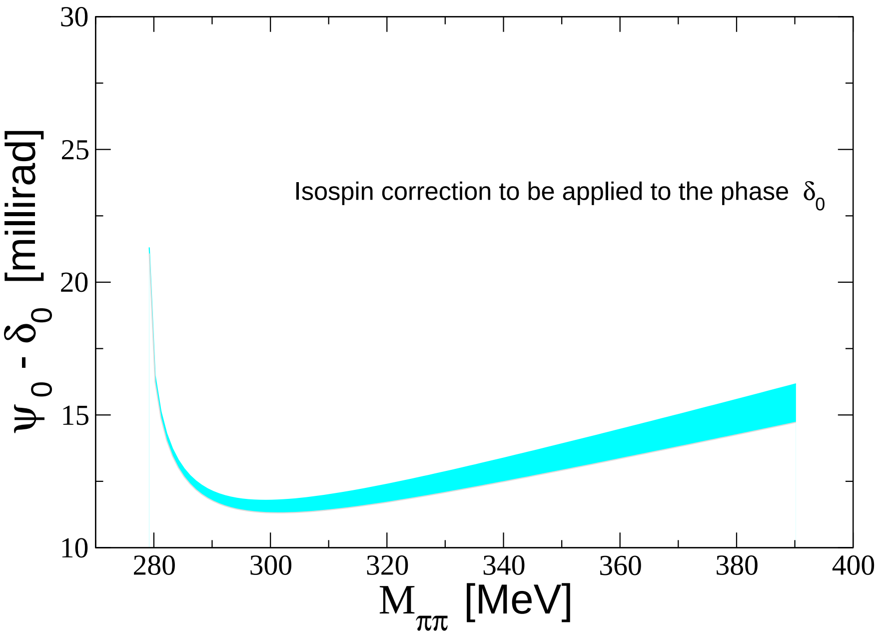


- generates isospin- breaking correction phase
- effect on the scattering length $a_0 \approx 10\%$!
- result: Bloch-Devaux, KAON 09

$$a_0 = 0.2209 \pm 0.0049(\text{stat}) \pm 0.0018(\text{syst}) \pm 0.0064(\text{th})$$



Pionium lifetime

- $\pi^+\pi^-$ system, bound by electromagnetism
  calculate energy levels as in quantum mechanics for the hydrogen atom!
- energy levels perturbed by strong interactions:
  ground state not stable, decays: $A_{\pi^+\pi^-} \to \pi^0\pi^0, \gamma\gamma, \ldots$
- (improved) Deser formula for the width: Deser 1954, Gasser et al. 2001

$$\begin{aligned}\Gamma &= \frac{2}{9}\alpha^3 p|\mathcal{A}(\pi^+\pi^- \to \pi^0\pi^0)_{\text{thr}}|^2(1+\epsilon) \\ &= \frac{2}{9}\alpha^3 p|a_0 - a_2|^2(1+\delta) \\ \delta &= 0.058 \pm 0.012\end{aligned}$$

- taking scattering length information from theory, predict lifetime:

$$\tau = (2.9 \pm 0.1) \times 10^{-15} s$$

- turn argument around: measure τ, extract $a_0 - a_2$ DIRAC 2005